Geometric aspects of higher order variational principles on submanifolds

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Abstract

The geometry of jets of submanifolds is studied, with special interest in the relationship with the calculus of variations. A new intrinsic geometric formulation of the variational problem on jets of submanifolds is given. Working examples are provided.

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Introduction

Jets of submanifolds (also known as *manifolds of contact elements*) are a natural framework for a geometric study of differential equations and the calculus of variations [1, 6, 7, 8, 14, 22, 29]. The space of r-th order jets of submanifolds $J^r(E, n)$ is introduced through the notion of contact of order r between n-dimensional submanifolds of a given manifold E. These spaces generalize jets of local sections in the sense that submanifolds which are not transversal to a fibration are also considered.

In this paper we devote ourselves to the calculus of variations on $J^r(E, n)$. This subject has been started in a modern framework in the pioneering papers [7, 28], where the $\mathcal{C}$-spectral sequence was introduced (see section 2). The main problem with respect to jets of fiberings is the absence of a distinguished space of independent variables. This complicates the computations of the terms of the $\mathcal{C}$-spectral sequence.

Let us make the above problem more clear with an example. An r-th order Lagrangian on a bundle $\pi\colon E \to M$ is a section $\lambda\colon J^r\pi \to \wedge^n T^*M$, where $J^r\pi$ is the r-th order jet of π and $n = \dim M$. This section can be regarded as an equivalence class $\lambda = [\alpha] \in \Lambda^n_r/\mathcal{C}^1\Lambda^n_r$, where Λ^n_r is the space of n-forms on $J^r\pi$ and $\mathcal{C}^1\Lambda^n_r$ is the subspace of 1-contact n forms, *i.e.* n-forms vanishing on (the r-order prolongation of) sections of π. A further property of such forms is that they yield no contribution to action-like

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functionals. The space $\Lambda_r^n/\mathcal{C}^1\Lambda_r^n$ is an element of the first term of the $\mathcal{C}$-spectral sequence. The problem is then: how to represent an object in $\Lambda_r^n/\mathcal{C}^1\Lambda_r^n$ in view of the absence of a distinguished space of independent variables?

This problem was first considered in [6], where it was proposed to use a sheaf of local Lagrangians whose difference on the intersection of neighborhoods was a contact form.

In this paper we discuss the use of the bundle of "truncated" total derivatives $H^{r+1,r} \to J^{r+1}(E, n)$ as a natural analogue of the bundle TM in the non-fibered case. We show that this bundle can be used to represent forms in the first term of the $\mathcal{C}$-spectral sequence. Note that the bundle $H^{1,0}$ was introduced in [13] with the purpose of studying geometric objects, *i.e.* tensor fields on submanifolds of a given manifold which depend on derivatives of the immersion, in homogeneous manifolds. Later on, the bundle $H^{1,0}$ appeared in [22] with the purpose of studying higher order analogues of connections on jets of submanifolds.

We use the bundle $H^{r+1,r}$ to represent Lagrangians, Euler–Lagrange and Helmholtz morphism so that we provide a geometric (*i.e.*, invariant, coordinate-free) formulation of variational problems on jets of submanifolds. Our formulation reduces to well-known formulations in the case of jets of fiberings (see for example [26]), and is a radical improvement of the old formulation by Dedecker [6] (see section 3 for details).

The theory is illustrated by two examples: the minimal submanifolds equation and the equation of relativistic particle motion. In the first example we study the calculus of variations on geometric objects arising in Riemannian geometry. An alternative geometric approach can be found in [3] in the framework of exterior differential systems. In the second example we consider the relativistic mechanics introduced in the framework of jets of submanifolds in [10] (see also [20, 32]) and discuss its variational formulation.

1   Jet spaces

Here we recall the main definitions used in the present paper. Our main sources are [1, 4, 14, 30]. By $J^r(E, n)$ we denote the r-jet of n-dimensional (immersed) submanifolds of an $(n + m)$-dimensional manifold E. We have the obvious projections

$$\cdots \to J^r(E, n) \overset{\pi_{r,r-1}}{\to} J^{r-1}(E, n) \to \cdots \to J^2(E, n) \overset{\pi_{2,1}}{\to} J^1(E, n) \overset{\pi_{1,0}}{\to} E,$$

whose inverse limit is the infinite order jet $J^\infty(E, n)$. We denote by $j_r L\colon L \to J^r(E, n)$, $p \mapsto [L]_p^r$, the prolongation of an n-dimensional submanifold $L \subset E$. We set $L^{(r)} \overset{\text{def}}{=} j_r L(L)$. *R-planes* are the tangent planes to r-th order prolonged submanifolds. For any $[L]_p^r \in J^r(E, n)$, R-planes passing through it biunivocally correspond to $(r+1)$-jets projecting on $[L]_p^r$: namely, $[L]_p^{r+1}$ corresponds to the tangent plane $R_{[L]_p^{r+1}} = T_{[L]_p^r} L^{(r)}$. The *Cartan subspace* $\mathcal{C}_\theta^r \subset T_\theta J^r(E, n)$ is defined as the span of all R-planes at the point θ, so that we have the *Cartan distribution* $\mathcal{C}^r\colon \theta \mapsto \mathcal{C}_\theta^r$ on $J^r(E, n)$. We have

$$(1) \qquad\qquad \mathcal{C}_\theta^r = R_{\overline{\theta}} \oplus \ker d_\theta \pi_{r,r-1}, \quad \overline{\theta} \in \pi_{r+1,r}^{-1}(\theta).$$

A diffeomorphism of $J^r(E,n)$ is called a *contact transformation* if it preserves $\mathcal{C}^r$. Any such transformation can be lifted to a jet space of higher order. A vector field on $J^r(E,n)$ whose local flow consists of contact transformations is called a *contact field*.

1 Remark. The space $J^r(E,n)$ can be also defined as the quotient of the higher order tangent bundle of regular (n,r)-velocities reg $T_n^r E$ (it is the set of r-jets of local immersions $\mathbb{R}^n \to E$ at 0, see [12]) with respect to the group G_n^r of r-jets of 'reparametrizations' [8, 9, 12, 17].

Notation: In what follows all manifolds and maps are smooth. Consider a manifold E, $\dim E = n + m$. Greek indexes λ, μ run from 1 to n and Latin indexes i, j run from 1 to m. Multiindexes will be denoted by the further Greek letters $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$, where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_k)$ with $1 \leq \sigma_j \leq n$ and $|\boldsymbol{\sigma}| \overset{\text{def}}{=} k \leq r$ (and analogously for $\boldsymbol{\tau}$). A *divided chart* on E is a chart of the form (x^λ, u^i), where $1 \leq \lambda \leq n$ and $1 \leq i \leq m$. Unless otherwise specified, a divided chart induces a chart on $J^r(E,n)$, that we shall denote by $(x^\lambda, u^i_{\boldsymbol{\sigma}})$. Einstein convention will be used.

1.1   Contact sequences

R-planes allow us to construct a short exact sequence of vector bundles over jets. This construction already appeared in [13] for the purpose of studying geometric objects on submanifolds and in [22] for the study of generalized connections on jets of submanifolds.

For $r \geq 0$, consider the following bundles over $J^{r+1}(E,n)$: the pull-back bundle

$$(2) \qquad T^{r+1,r} \overset{\text{def}}{=} J^{r+1}(E,n) \underset{J^r(E,n)}{\times} TJ^r(E,n),$$

the subbundle $H^{r+1,r}$ of $T^{r+1,r}$ defined by

$$(3) \qquad H^{r+1,r} \overset{\text{def}}{=} \left\{ \left([L]^{r+1}_p, \upsilon\right) \in T^{r+1,r} \mid \upsilon \in R_{[L]^{r+1}_p} \right\},$$

and the quotient bundle $V^{r+1,r} \overset{\text{def}}{=} T^{r+1,r}/H^{r+1,r}$. The bundles $H^{r+1,r}$ and $V^{r+1,r}$ are strictly related with the horizontal and vertical bundles in the case of jets of fiberings (see remark 3).

2 Definition. We call $H^{r+1,r}$ and $V^{r+1,r}$, respectively, the *pseudo-horizontal* and the *pseudo-vertical* bundle of $J^r(E,n)$.

The pseudo-horizontal bundle has some additional features. The following isomorphism over $\mathrm{id}_{J^{r+1}(E,n)}$ holds:

$$(4) \qquad H^{r+1,r} \to J^{r+1}(E,n) \times_{J^1(E,n)} H^{1,0}, \quad \left([L]^{r+1}_p, \upsilon\right) \mapsto \left([L]^{r+1}_p, d\pi_{r,0}(\upsilon)\right).$$

The restriction of $H^{r+1,r}$ to an n-dimensional submanifold $L \subset E$ is isomorphic to TL, as it is easily seen. By definition we have the following *contact exact sequence*

$$(5) \qquad 0 \to H^{r+1,r} \overset{D}{\hookrightarrow} T^{r+1,r} \overset{\omega}{\to} V^{r+1,r} \to 0,$$

where D and ω are, respectively, the natural inclusion and quotient projection.

3 Remark. If E has a fiber structure $\pi : E \to M$, then $H^{r+1,r}$ is isomorphic to $J^{r+1}\pi \times_M TM$ (the horizontal bundle), and $V^{r+1,r}$ to $J^{r+1}\pi \times_{J^r\pi} \ker d\pi_r$, where $\pi_r = \pi \circ \pi_{1,0} \circ \cdots \circ \pi_{r,r-1}$ (the vertical bundle), so that the above contact sequence splits.

Let us evaluate the coordinate expressions of D and ω. A local basis of the space of sections of the bundle $H^{r+1,r}$ is

$$D_\lambda = \frac{\partial}{\partial x^\lambda} + u^j_{\boldsymbol{\sigma},\lambda} \frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}}, \quad |\boldsymbol{\sigma}| \leq r$$

where the index $\boldsymbol{\sigma}, \lambda$ stands for $(\sigma_1, \ldots, \sigma_s, \lambda)$ with $s \leq r$. A local basis of the space of sections $(H^{r+1,r})^*$ dual to $\{D_\lambda\}$ is given by the restriction of the 1-forms dx^λ to $H^{r+1,r}$, and is denoted by $\overline{dx^\lambda}$. The local expression of D turns out to be $D = \overline{dx^\lambda} \otimes D_\lambda$. A local basis of the space of sections of the bundle $V^{r+1,r}$ is

$$B^{\boldsymbol{\sigma}}_j \overset{\text{def}}{=} \left[\frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}} \right] , \qquad |\boldsymbol{\sigma}| \leq r.$$

The local expression of ω turns out to be $\omega = \omega^j_{\boldsymbol{\sigma}} \otimes B^{\boldsymbol{\sigma}}_j$, where $\omega^j_{\boldsymbol{\sigma}} = du^j_{\boldsymbol{\sigma}} - u^j_{\boldsymbol{\sigma},\lambda} dx^\lambda$.

Let $VJ^1(E,n) \overset{\text{def}}{=} \ker d\pi_{1,0}$. We have the following

4 Lemma. $VJ^1(E,n) \simeq (H^{1,0})^* \otimes_{J^1(E,n)} V^{1,0}$.

The following theorem has been proved in [23], and in [18] with the help of previous lemma.

5 Theorem. *For $r \geq 1$, the bundles $\pi_{r+1,r}$ are affine bundles associated with the vector bundles $(\odot^{r+1}(H^{1,0})^*) \otimes_{J^r(E,n)} V^{1,0}$, where $\odot$ stands for the symmetric tensor product.*

In the case $r = 0$, $\pi_{1,0}$ coincides with the Grassmann bundle of n-dimensional subspaces in TE: in fact $\mathrm{Gr}(T_pE, n) \simeq \pi^{-1}_{1,0}(p)$. This contrasts with the case of jets of fiberings. Now, let (x^λ, u^i) and (y^μ, v^j) be two coordinate charts on $L \subset E$; let us denote by $(J^\mu_\lambda, J^\mu_i, J^j_\lambda, J^j_i)$ the Jacobian of the change of coordinates. Then the fibered coordinate change is given by the following formula

(6) $$v^j_\mu = A^\lambda_\mu (J^j_\lambda + J^j_i u^i_\lambda).$$

where $A^\lambda_\mu \overset{\text{def}}{=} (J^\mu_\lambda + J^\mu_i u^i_\lambda)^{-1}$.

1.2   Forms and vector fields on jets

Here we study the spaces of forms on finite order jets, and in particular *contact* and *horizontal* forms, in view of a geometrical formulation of variational principles.

We denote by $\mathcal{F}_r$ the algebra $C^\infty(J^r(E,n))$. For $k \geq 0$ we denote by Λ^k_r the $\mathcal{F}_r$-module of k-forms on $J^r(E,n)$, and by $\chi(J^r(E,n))$ the $\mathcal{F}_r$-module of vector fields on $J^r(E,n)$. We also set $\Lambda^*_r = \bigoplus_k \Lambda^k_r$. We introduce the submodule of Λ^k_r of the *contact forms* of order r

$$\mathcal{C}^1\Lambda^k_r \overset{\text{def}}{=} \{ \alpha \in \Lambda^k_r \mid (j_rL)^*\alpha = 0 \quad \text{for each } n\text{-dim. submanifold } L \subset E \}.$$

We set $\mathcal{C}^1\Lambda_r^* = \bigoplus_k \mathcal{C}^1\Lambda_r^k$. Moreover, we define $\mathcal{C}^p\Lambda_r^*$ as the p-th exterior power of $\mathcal{C}^1\Lambda_r^*$. Of course, $\mathcal{C}^p\Lambda_r^k = \mathcal{C}^p\Lambda_r^* \cap \Lambda_r^k$. We also consider the $\mathcal{F}_{r+1}$-module $\Lambda_{r+1,r}^k$ of sections of the bundle $\bigwedge^k(T^{r+1,r})^*$, the $\mathcal{F}_{r+1}$-module $\mathcal{H}_{r+1,r}^k$ of *pseudo-horizontal k-forms*, *i.e.*, sections of the bundle $\bigwedge^k(H^{r+1,r})^*$, and the $\mathcal{F}_{r+1}$-module of *pseudo-vertical* fields $\mathcal{V}^{r+1,r}$, *i.e.*, sections of $V^{r+1,r}$.

6 Definition. Let $q \in \mathbb{N}$. *Horizontalisation* is the map

$$h^{0,q} \colon \Lambda_r^q \to \mathcal{H}_{r+1,r}^q, \quad \alpha \mapsto \wedge^q(D^*) \circ (\pi_{r+1,r}^*\alpha).$$

Dually, *verticalization* is the map $v \colon \chi(J^r(E,n)) \to \mathcal{V}^{r+1,r}$, $v(X) \overset{\text{def}}{=} \omega \circ X \circ \pi_{r+1,r}$.

If $\alpha \in \Lambda_r^q$, then we have the coordinate expression

$$\text{(7)} \qquad \alpha = \alpha_{i_1\dots i_h}^{\boldsymbol{\sigma}_1\dots\boldsymbol{\sigma}_h}{}_{\lambda_{h+1}\dots\lambda_q} du_{\boldsymbol{\sigma}_1}^{i_1} \wedge \cdots \wedge du_{\boldsymbol{\sigma}_h}^{i_h} \wedge dx^{\lambda_{h+1}} \wedge \cdots \wedge dx^{\lambda_q},$$

where $0 \le h \le q$ and $0 \le |\boldsymbol{\sigma}| \le r$. Hence

$$\text{(8)} \qquad h^{0,q}(\alpha) = u_{\boldsymbol{\sigma}_1,\lambda_1}^{i_1} \cdots u_{\boldsymbol{\sigma}_h,\lambda_h}^{i_h} \alpha_{i_1\dots i_h}^{\boldsymbol{\sigma}_1\dots\boldsymbol{\sigma}_h}{}_{\lambda_{h+1}\dots\lambda_q} \overline{dx}^{\lambda_1} \wedge \cdots \wedge \overline{dx}^{\lambda_q}.$$

If $X = a^\lambda\partial/\partial x^\lambda + b_{\boldsymbol{\sigma}}^j\partial/\partial u_{\boldsymbol{\sigma}}^j \in \chi(J^r(E,n))$, then $v(X) = (b_{\boldsymbol{\sigma}}^j - a^\lambda u_{\boldsymbol{\sigma},\lambda}^j)B_j^{\boldsymbol{\sigma}}$.

Let us introduce the $\mathcal{F}_r$-module $\bar{\Lambda}_r^q \overset{\text{def}}{=} \operatorname{im} h^{0,q}$.

7 Theorem. *The space $\bar{\Lambda}_r^q$ consists of elements of $\mathcal{H}_{r+1,r}^q$ whose coefficients are fibered polynomials of degree q in the highest order variables (i.e., $u_{\boldsymbol{\sigma}_k,\lambda_k}^{i_k}$ with $|\boldsymbol{\sigma}_k| = r$).*

Proof. The property of the statement holds in any chart from the coordinate expression (8). It remains to prove that this is an intrinsic property.

For $r \ge 1$ the property is intrinsic due to the affine structure of $\pi_{r+1,r}$.

For $r = 0$, in a similar way to (6), we realize that $\overline{dy}^\mu = (J_\lambda^\mu + J_i^\mu u_\lambda^i)\overline{dx}^\lambda$. Combining this formula with (8) we deduce that the set of sections of the bundle $\bar{\Lambda}_0^q$ admit a subspace of sections with polynomial coefficients, even if $\pi_{1,0}$ *is not* an affine bundle. $\square$

Note that $\bar{\Lambda}_r^q$ does not coincide with the space of *all* such polynomial forms unless $n = 1$. In fact, (8) shows that the coefficients of monomials are skew-symmetric w.r.t. the exchange of pairs $\overset{i_j}{\boldsymbol{\sigma}_j}$ and $\overset{i_k}{\boldsymbol{\sigma}_k}$. This property appears in an analogous way in the case of jets of fiberings, see [2]. Since D is the identity on $TL^{(r)}$, we have the following

8 Lemma. *Let $\alpha \in \Lambda_r^q$, with $0 \le q \le n$. We have $(j_rL)^*(\alpha) = (j_{r+1}L)^*(h^{0,q}(\alpha))$.*

As an obvious consequence of the previous lemma, we have:

$$\text{(9)} \qquad \mathcal{C}^1\Lambda_r^q = \ker h^{0,q} \quad \text{if} \quad 0 \le q \le n, \qquad \mathcal{C}^1\Lambda_r^q = \Lambda_r^q \quad \text{if} \quad q > n.$$

Moreover, $\alpha \in \mathcal{C}^1\Lambda_r^q$ if and only if $\pi_{r+1,r}^*(\alpha) \in \operatorname{im}((\omega)^* \wedge \operatorname{id})$. Hence, if $\alpha \in \mathcal{C}^p\Lambda_r^{p+q}$, then

$$\text{(10)} \qquad \pi_{r+1,r}^*(\alpha) = \omega_{\boldsymbol{\sigma}_1}^{i_1} \wedge \cdots \wedge \omega_{\boldsymbol{\sigma}_p}^{i_p} \wedge \alpha_{i_1\dots i_p}^{\boldsymbol{\sigma}_1\dots\boldsymbol{\sigma}_p}, \qquad \alpha_{i_1\dots i_p}^{\boldsymbol{\sigma}_1\dots\boldsymbol{\sigma}_p} \in \pi_{r+1,r}^*\left(\Lambda_r^q\right),$$

where $|\boldsymbol{\sigma}_l| \leq r$ for $l = 1, \ldots, p$. Note that, if $q = 0$, then $|\boldsymbol{\sigma}_l| \leq r - 1$ for $l = 1, \ldots, p$. Moreover, derivatives of order $r + 1$ appear in the above expression when $|\boldsymbol{\sigma}_l| = r$. It is possible to obtain an expression containing just r-th order derivatives by using contact forms of the type $d\omega_{\boldsymbol{\sigma}_l}^{i_l}$ with $|\boldsymbol{\sigma}_l| = r - 1$ (see [16]).

For future purposes, we introduce the following *partial horizontalisation* map

(11) $$ h^{p,q} \colon \Lambda_r^{p+q} \to \Lambda_{r+1,r}^p \otimes \bar{\Lambda}_r^q, \quad \alpha \mapsto (\wedge^p \mathrm{id} \otimes \wedge^q D^*) \circ (\pi_{r+1,r}^* \alpha). $$

The action of $h^{p,q}$ on decomposable forms is

$$ h^{p,q}(\alpha_1 \wedge \ldots \wedge \alpha_{p+q}) = $$
$$ \frac{1}{p!\,q!} \sum_{\varsigma \in S_{p+q}} |\varsigma| \, \pi_{r+1,r}^*(\alpha_{\varsigma(1)} \wedge \ldots \wedge \alpha_{\varsigma(p)}) \otimes h^{0,q}(\alpha_{\varsigma(p+1)} \wedge \ldots \wedge \alpha_{\varsigma(p+q)}), $$

where S_{p+q} is the set of permutations of $p + q$ elements.

2 $\mathcal{C}$-spectral sequence and variational sequence

In this section we show that the terms of the $\mathcal{C}$-spectral sequence can be explicitly computed through the pseudo-horizontal and pseudo-vertical bundles and the horizontalization.

Let $r > 0$. We have the *bounded* filtration of modules

(12) $$ \Lambda_r^k \overset{\mathrm{def}}{=} \mathcal{C}^0 \Lambda_r^k \supset \mathcal{C}^1 \Lambda_r^k \supset \cdots \supset \mathcal{C}^p \Lambda_r^k \supset \cdots \supset \mathcal{C}^I \Lambda_r^k \supset \mathcal{C}^{I+1} \Lambda_r^k = \{0\} $$

(see [16] for the value of I). This is a differential filtration, *i.e.* $d(\mathcal{C}^p \Lambda_r^k) \subset \mathcal{C}^p \Lambda_r^{k+1}$, hence it gives rise to a spectral sequence $(E_k^{p,q}, d_k^{p,q})$, with $k, p, q \geq 0$, which we call the $\mathcal{C}$-*spectral sequence of (finite) order* r on E. By construction, it is invariant w.r.t. contact transformations of $J^r(E, n)$.

We recall that $E_0^{p,q} \equiv \mathcal{C}^p \Lambda_r^{p+q} / \mathcal{C}^{p+1} \Lambda_r^{p+q}$. We have the following simple result.

9 Proposition. *The restriction of $h^{p,q}$ to $\mathcal{C}^p \Lambda_r^{p+q}$ yields the injective morphism*

$$ E_0^{p,q} \to \Lambda_{r+1,r}^p \otimes \bar{\Lambda}_r^q, \quad [\alpha] \mapsto h^{p,q}(\alpha). $$

It follows that $\bar{d}(h^{p,q}(\alpha)) = h^{p,q+1}(d\alpha)$, where $\bar{d} \overset{\mathrm{def}}{=} d_0^{p,q}$.

The finite order $\mathcal{C}$-spectral sequence is filtered with respect to the order r: pull-back through $\pi_{r+1,r}$ provides an injective morphism between the $\mathcal{C}$-spectral sequences of order r and $r+1$, respectively. It follows that the infinite order $\mathcal{C}$-spectral sequence [4, 28, 29] can be obtained as the direct limit of the finite order $\mathcal{C}$-spectral sequences. Until the end of this section we use the infinite order formulation for the sake of simplicity. Below we denote by $\mathcal{F}$ the direct limit of $\mathcal{F}_r$, and $\bar{\Lambda}^n$ the direct limit of $\bar{\Lambda}_r^n$. Note that $\alpha \in \bar{\Lambda}^n$ has the coordinate expression $\lambda = \lambda_0 \mathcal{V}_n$, where $\lambda_0 \in \mathcal{F}_r$ for some r and $\mathcal{V}_n \overset{\mathrm{def}}{=} n! \, \overline{dx}^1 \wedge \cdots \wedge \overline{dx}^n$ is a local volume form on any submanifold.

For $0 \leq q \leq n$ and $p \geq 1$ the terms $E_0^{0,q}$ and $E_1^{p,n}$ may be arranged into a further complex which is of fundamental importance for the calculus of variations (see also the end of section 3): the variational sequence.

10 Definition. The complex

$$\ldots \xrightarrow{\bar{d}} \bar{\Lambda}^{n-1} \xrightarrow{\bar{d}} \bar{\Lambda}^n \xrightarrow{\mathcal{E}} E_0^{1,n}/\bar{d}(E_0^{1,n-1}) \xrightarrow{\mathcal{H}} E_0^{2,n}/\bar{d}(E_0^{2,n-1}) \xrightarrow{d_1^{2,n}} \ldots,$$

where $\mathcal{E}$ is the composition of the projection $\bar{\Lambda}^{0,n} \to E_1^{0,n} = \bar{\Lambda}^{0,n}/\bar{d}(\bar{\Lambda}^{0,n-1})$ with $d_1^{0,n}$ and $\mathcal{H} \stackrel{\text{def}}{=} d_1^{1,n}$, is said to be the *variational sequence*. Elements of $E_0^{p,n}/\bar{d}(E_0^{p,n-1})$ are said to be *variational p-forms*.

It is easy to prove that $\bar{d}\bar{\alpha} = D_\lambda(\alpha^{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_l}_{j_1\cdots j_l\,\lambda_{l+1}\cdots\lambda_q})u^{j_1}_{\boldsymbol{\tau}_1,\lambda_1}\cdots u^{j_l}_{\boldsymbol{\tau}_l,\lambda_l}\overline{dx}^\lambda \wedge \overline{dx}^{\lambda_1} \wedge \cdots \wedge \overline{dx}^{\lambda_q}$ for $|\boldsymbol{\tau}_h| = r$. This expression is similar to that of the horizontal differential in jets of fiberings, but the presence of $\overline{dx}^\lambda$ yields different transformation properties.

Now we shall prove that any variational form can be represented by a distinguished object. To this purpose, the most important tool is Green's formula [27], which is the geometric analogue of the integration by parts.

Let P, Q be modules of local sections of vector bundles on $J^\infty(E,n)$. We recall that a $\mathcal{C}$-*differential operator* $\Delta\colon P \to Q$ is a linear differential operator which admits a restriction on prolonged submanifolds $L^{(\infty)}$. In coordinates, we have $\Delta = (a_{ij}^{\boldsymbol{\sigma}}D_{\boldsymbol{\sigma}})$, where $a_{ij}^{\boldsymbol{\sigma}} \in \mathcal{F}$, $D_{\boldsymbol{\sigma}} = D_{\sigma_1} \circ \cdots \circ D_{\sigma_h}$, and $|\boldsymbol{\sigma}| = h \leq k$. The space of such operators is denoted by $\mathcal{C}\mathrm{Diff}_k(P,Q)$. We shall also deal with spaces of antisymmetric $\mathcal{C}$-differential operators of l arguments in P, which we denote by $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(l)\,k}(P,Q)$.

Now, let $\hat{P} \stackrel{\text{def}}{=} \mathrm{Hom}(P, \bar{\Lambda}^n)$; we say $\hat{P}$ is the *variational dual* of P. Note that the variational dual is not exactly the same as in [29] because we use our 'concrete' space $\bar{\Lambda}^n$ instead of just the quotient $\Lambda^n/\mathcal{C}^1\Lambda^n$.

Let $\Delta \in \mathcal{C}\mathrm{Diff}(P,Q)$. Then, according with Green's formula [27, 4] (see [15, p. 31] for more details) there exists a unique $\Delta^* \in \mathcal{C}\mathrm{Diff}(\hat{Q}, \hat{P})$ such that

$$(13) \qquad\qquad \widehat{q}(\Delta(p)) - (\Delta^*(\widehat{q}))(p) = \bar{d}\omega_{p,\widehat{q}}(\Delta)$$

for all $\widehat{q} \in \widehat{Q}$, $p \in P$, where $\omega_{p,\widehat{q}}(\Delta) \in \bar{\Lambda}^{n-1}$. In coordinates, if $\Delta(\varphi)^i = \Delta_j^{\boldsymbol{\sigma} i}D_{\boldsymbol{\sigma}}\varphi^j$, then $\Delta^*(\psi)_j = (-1)^{|\boldsymbol{\sigma}|}D_{\boldsymbol{\sigma}}(\Delta_j^{\boldsymbol{\sigma} i}\psi_i)$.

Let us introduce the $\mathcal{F}$-module $\varkappa$ of local sections of $J^\infty(E,n) \times_{J^1(E,n)} V^{1,0}$. With an element $\varphi \in \varkappa$, a non-trivial symmetry $\Im_\varphi$ of the Cartan distribution on $J^\infty(E,n)$ (*i.e.* a symmetry which is not contained in the distribution) is associated [4]. Locally, if $\varphi = \varphi^i B_i$, then $\Im_\varphi = D_{\boldsymbol{\sigma}}(\varphi^i)B_i^{\boldsymbol{\sigma}}$. Moreover, let $K_p(\varkappa) \subset \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}(\varkappa, \hat{\varkappa})$ be the subspace of operators which are skew-adjoint in each argument. With a reasoning similar to the fibered case [4, p. 190] we obtain the isomorphisms

$$\nabla\colon E_0^{p,q} \to \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa, \bar{\Lambda}^q), \quad \nabla_\beta(\varphi_1,\ldots,\varphi_p) = \frac{1}{p!}\,\Im_{\varphi_1}\lrcorner\left(\cdots\lrcorner\Im_{\varphi_p}\lrcorner\beta\cdots\right)$$

$$I_p\colon E_0^{p,n}/\bar{d}E_0^{p,n-1} \to K_p(\varkappa), \quad I_p([\Delta])(\varphi_1,\ldots,\varphi_{p-1})(\varphi_p) \stackrel{\text{def}}{=} \Delta(\varphi_1,\ldots,\varphi_{p-1},\cdot)^*(1)(\varphi_p),$$

where we set $\beta = h^{p,q}(\alpha)$ and $\Delta = \nabla_\beta$ for $\alpha \in \mathcal{C}^p\Lambda^{p+q}$. We have the coordinate expressions

$$(14) \qquad \Delta(\varphi_1,\ldots,\varphi_p) = \Delta^{\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_{p-1}\boldsymbol{\tau}}_{i_1\cdots i_{p-1}j}D_{\boldsymbol{\sigma}_1}(\varphi_1^{i_1})\cdots D_{\boldsymbol{\sigma}_{p-1}}(\varphi_{p-1}^{i_{p-1}})D_{\boldsymbol{\tau}}(\varphi_p^j)\,\mathcal{V}_n,$$

$$(15) \qquad I_p([\Delta])(\varphi_1,\ldots,\varphi_p) = (-1)^{|\boldsymbol{\tau}|}D_{\boldsymbol{\tau}}(\Delta^{\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_{p-1}\boldsymbol{\tau}}_{i_1\cdots i_{p-1}j}D_{\boldsymbol{\sigma}_1}(\varphi_1^{i_1})\cdots D_{\boldsymbol{\sigma}_{p-1}}(\varphi_{p-1}^{i_{p-1}}))\varphi_p^j\,\mathcal{V}_n,$$

where $0 \leq |\boldsymbol{\sigma}_1|, \ldots, |\boldsymbol{\sigma}_{p-1}|, |\boldsymbol{\tau}| \leq r$. Note that if $\alpha \in \mathcal{C}^p \Lambda_r^{p+q}$ then $\Delta_{i_1 \ldots i_{p-1} j}^{\boldsymbol{\sigma}_1 \ldots \boldsymbol{\sigma}_{p-1} \boldsymbol{\tau}}$ is an n-th degree polynomial in the highest order derivatives with the structure described in theorem 7. A coordinate expression of $\mathcal{E}$ can be obtained as follows. Let $\lambda \in \bar{\Lambda}^n$, with $\lambda = \lambda_0 \mathcal{V}_n$; then the form $\alpha = \lambda_0 n! \, dx^1 \wedge \cdots \wedge dx^n$ fulfills $h^{0,n}(\alpha) = \lambda$, hence, by definition, $d_1^{0,n}([\lambda]) = ([[d\alpha]]) = [h^{1,n}(d\alpha)]$. So,

$$(16) \qquad \mathcal{E}(\lambda)(\varphi_1) = (-1)^{|\boldsymbol{\tau}|} D_{\boldsymbol{\tau}} \left(\frac{\partial}{\partial u_{\boldsymbol{\tau}}^i} \lambda_0 \right) \varphi_1^i \, \mathcal{V}_n \, .$$

In an analogous way we obtain the coordinate expression for the case $p > 1$.

3   Variational principles

In this section we formulate higher-order variational principles on submanifolds in the 'classical' way, *i.e.*, with an integral formalism. The main difference with the fibered case is the absence of a space of independent variables. We get rid of this difficulty by the pseudo-horizontal bundle and the horizontalization operator. A similar approach was already attempted in [6] where the author was forced to use families of Lagrangians defined on open subsets with the property that, on intersecting subsets, the action be the same. In what follows, the pseudo-horizontal bundle allows us to use single objects as Lagrangians. More precisely, a Dedecker family of Lagrangians yields just one bundle morphism $\lambda \colon J^r(E, n) \to \wedge^n (H^{1,0})^*$. The Euler–Lagrange equations follow straightforwardly, as we will see.

11 Definition. A form $h^{0,n}(\alpha) = \lambda \in \bar{\Lambda}_r^{0,n}$ is said to be an *r-th order (generalized) Lagrangian*. The *action* of the Lagrangian λ on an n-dimensional oriented submanifold $L \subset E$ with compact closure and regular boundary is defined by

$$(17) \qquad \int_L (j_r L)^* \alpha.$$

The word 'generalized' comes from the fact that λ depends on $(r+1)$-st derivatives in the way specified in equation (8). Moreover, the action is well-defined because only the horizontal part of a form α contributes to the action (lemma 8). Now, we formulate the variational problem. Let $L \subset E$ be as in the above definition. A vector field X on E vanishing on ∂L is called a *variation field*. The submanifold L is *critical* if for each variation field X with flow ϕ_t we have

$$(18) \qquad \frac{d}{dt}\Big|_{t=0} \int_L (\phi_t^{(r)} \circ j_r L)^* \alpha = 0$$

where we recall that $\phi_t^{(r)} \colon J^r(E, n) \longrightarrow J^r(E, n)$ is the r-lift of ϕ_t (see section 1). In what follows by $X^{(r)}$ we denote the r-lift of a vector field X on E.

12 Lemma. *For any $\alpha \in \mathcal{C}^1 \Lambda^{q+1}$, we have that $h^{0,q}(X^{(r)} \lrcorner \alpha) = v(X^{(r)}) \lrcorner (h^{1,q}(\alpha))$.*

Proof. In fact,

$$
\begin{aligned}
h^{0,q}\left(X^{(r)}\lrcorner\alpha\right) &= \wedge^q D^*\left(\pi_{r+1,r}^*\left(X^{(r)}\lrcorner\alpha\right)\right) = \wedge^q D^*\left(X^{(r)}\lrcorner\left(\pi_{r+1,r}^*\alpha\right)\right) \\
&= \wedge^q D^*\left(X^{(r)}\lrcorner\left(\omega_{\boldsymbol{\sigma}}^i\wedge\beta_i^{\boldsymbol{\sigma}}\right)\right) = \wedge^q D^*\left(\omega_{\boldsymbol{\sigma}}^i(v(X^{(r)}))\beta_i^{\boldsymbol{\sigma}}\right) \\
&= \left((\mathrm{id}\otimes\wedge^q D^*)\left(\omega_{\boldsymbol{\sigma}}^i\wedge\beta_i^{\boldsymbol{\sigma}}\right)\right)v(X^{(r)}) = v(X^{(r)})\lrcorner\left(h^{1,q}\alpha\right).
\end{aligned}
$$

$\square$

13 Lemma. *For any $\beta\in E_0^{1,q}$, we have that $v(X^{(r)})\lrcorner(\bar{d}\beta)+\bar{d}(v(X^{(r)})\lrcorner\beta)=0$.*

Proof. Let $\alpha\in\mathcal{C}^1\Lambda_r^{1+q}$ be such that $h^{1,q}(\alpha)=\beta$. We have that

$$
v(X^{(r)})\lrcorner\left(\bar{d}(h^{1,q}(\alpha))\right) = v(X^{(r)})\lrcorner\left(h^{1,q+1}(d\alpha)\right)
$$

and in view of lemma 12, the above right side term is equal to $h^{0,q+1}\left(X^{(r)}\lrcorner d\alpha\right)$. Furthermore, taking into account that $\mathcal{L}_{X^{(r)}}\alpha=X^{(r)}\lrcorner d\alpha+d(X^{(r)}\lrcorner\alpha)$ is a 1-contact form, we have that

$$
h^{0,q+1}\left(X^{(r)}\lrcorner d\alpha\right) = -h^{0,q+1}\left(d(X^{(r)}\lrcorner\alpha)\right) = -\bar{d}\left(h^{0,q}(X^{(r)}\lrcorner\alpha)\right) = -\bar{d}\left(v(X^{(r)})\lrcorner h^{1,q}(\alpha)\right)
$$

where the last equality is attained by applying again lemma 12.

$\square$

14 Theorem. *Let $\lambda\in\bar{\Lambda}_r^n$. Then an (embedded) submanifold $L\subset E$ is critical for λ if and only if the following Euler–Lagrange equations are fulfilled:*

$$
\tag{19} \mathcal{E}(\lambda)\circ j_{2r}L = 0.
$$

Proof. We show that (18) depends on the vertical part $v(X)$ of X, and provide the Euler–Lagrange equations. We have

$$
\tag{20}
\begin{aligned}
\frac{d}{dt}\bigg|_{t=0}\int_L(\phi^{(r)}\circ j_rL)^*\alpha &= \int_L(j_rL)^*\mathcal{L}_{X^{(r)}}\alpha = \int_L(j_rL)^*X^{(r)}\lrcorner d\alpha \\
&= \int_L(j_{r+1}L)^*h^{0,n}(X^{(r)}\lrcorner d\alpha) = \int_L(j_{r+1}L)^*v(X^{(r)})\lrcorner h^{1,n}(d\alpha) = \int_L(j_{2r+1}L)^*v(X)\lrcorner\mathcal{E}(\lambda).
\end{aligned}
$$

Here we used Stokes' theorem, lemma 8 and lemma 12, taking into account that $d\alpha\in\Lambda_r^{n+1}=\mathcal{C}^1\Lambda_r^{n+1}$, up to the final equality. Green's formula yields

$$
\tag{21} v(X^{(r)})\lrcorner h^{1,n}(d\alpha) = \mathcal{E}(\lambda)(v(X))+\bar{d}\omega, \quad \omega\in\bar{\Lambda}^{n-1},
$$

hence the last equality of (20) follows from the identity $(j_{r+1}L)^*\bar{d}\omega=d(j_{r+1}L)^*\omega$, and Stokes' theorem. By virtue of the fundamental lemma of calculus of variations, equation (18) vanishes if and only if $(j_{2r+1}L)^*\mathcal{E}(\lambda)=0$, which is equivalent to the Euler–Lagrange equation (19).

$\square$

Note that, in view of lemma 13, we have $h^{1,n}(d\alpha) = \mathcal{E}(\lambda) + \bar{d}\omega$, with $\omega \in E_0^{1,n-1}$.

It is now clear that in the variational sequence $E_1^{1,n}$ is the space of *Euler–Lagrange type morphism*, and $\mathcal{E}$ takes a Lagrangian λ into its Euler–Lagrange form $\mathcal{E}(\lambda)$. If $\mathcal{E}(\lambda) = 0$ then the Lagrangian is trivial, or *null*.

We could continue our analysis and show that $E_1^{2,n}$ is the space of *Helmholtz type morphism*. The operator $\mathcal{H}$ takes an Euler–Lagrange type form ϵ into its Helmholtz form $\mathcal{H}(\epsilon)$. If $\mathcal{H}(\epsilon) = 0$, or, equivalently, if ϵ is *locally variational*, then ϵ comes from a local Lagrangian. In other words, the associated differential equation $\epsilon \circ j_r L = 0$ is locally variational.

15 Remark. By construction, and in view of Lie–Bäcklund theorem [4], the Euler–Lagrange operator $\mathcal{E}$ and the Helmholtz operator $\mathcal{H}$ are invariant w.r.t. contact transformations. In particular, if $m > 1$ then they are invariant w.r.t. point transformations.

16 Remark. In the parametric approach [5] the variational principle is formulated on reg $T_n^r E$ under the hypothesis that the Lagrangian commute with the action of the group of parametrizations (remark 1). This leads to extra computations in order to verify at each step the invariance of objects with respect to changes of parametrization.

4   Examples

In this section we will present two examples, one from differential geometry (the equation of minimal submanifolds) and the other from mathematical physics (the equation of particle motion in general relativity).

4.1   The minimal submanifold equation

Here we will show that the geometry of submanifolds of a given Riemannian manifold can be reformulated on jets of submanifolds. In fact, tensors that are defined on one submanifold and depend on derivatives of the immersion are converted into objects that are defined on jets of submanifolds (*geometric objects*, see [13] for the case when E is an homogeneous manifold).

Let (E, g) be a Riemannian manifold. Let us denote by Γ the Levi–Civita connection associated with g. The metric g can be lifted as a fiber metric on $T^{1,0}$ by composition with the projection $\pi_{1,0}$. We indicate the above metric on $T^{1,0}$ with the same letter g. We also have the contravariant metric $\bar{g}$ on $(T^{1,0})^*$. Let us set $V_g^{1,0} \stackrel{\text{def}}{=} (H^{1,0})^\perp$; note that the projection ω is an isomorphism between $V_g^{1,0}$ and $V^{1,0}$. We have the splitting $T^{1,0} = H^{1,0} \oplus V_g^{1,0}$. Note that, if $L \subset E$ is an n-dimensional submanifold of E, then the restriction to L of the bundle $V_g^{1,0}$ is just the normal bundle $N_g L$ to the submanifold L with respect to the metric g, as it can be easily seen.

The metric g restricts to the metric g^H on $H^{1,0}$, with coordinate expression

$$(22) \qquad g^H = g_{\lambda\mu}^H \overline{dx}^\lambda \otimes \overline{dx}^\mu = (g_{\lambda\mu} + g_{\lambda j} u_\mu^j + g_{i\mu} u_\lambda^i + g_{ij} u_\lambda^i u_\mu^j)\, \overline{dx}^\lambda \otimes \overline{dx}^\mu.$$

The above metric g^H can be characterized as follows. If $L \subset E$ is an n-dimensional submanifold of E, then $(j^1 L)^* g^H$ is the pull-back metric of g on L. For this reason we say g^H is the *universal first fundamental form* on n-dimensional submanifolds of E.

The contravariant metric $\bar{g}^H$ will be used; its coordinate expression is denoted by $\bar{g}^H = (\bar{g}^H)^{\lambda\mu} D_\lambda \otimes D_\mu$. We introduce the local basis $N_i \overset{\text{def}}{=} \partial/\partial u^i - (g_{\mu i} + g_{ij} u_\mu^j)(\bar{g}^H)^{\mu\lambda} D_\lambda$ of $V_g^{1,0}$. The metric g restricts to the metric g^V on $V_g^{1,0}$, with coordinate expression

$$g_{ij}^V = g(N_i, N_j) = g_{ij} - (g_{\lambda i} + g_{ik} u_\lambda^k)(g_{\mu j} + g_{jk} u_\mu^k)(\bar{g}^H)^{\lambda\mu}.$$

Let us introduce the operator
(23)
$$\mathbf{II} \colon J^2(E, n) \to (H^{1,0})^* \underset{J^2(E,n)}{\otimes} (H^{1,0})^* \underset{J^2(E,n)}{\otimes} V^{1,0}, \quad \mathbf{II}(X, Y, \varrho) = \varrho\left(\left[\pi_{2,0}^*(\Gamma) \right]_{\widetilde{X}} (Y) \right)$$

where $\widetilde{X}$ is a field on $J^2(E, n)$ lying in the Cartan distribution and projecting on X (see (1)), and $[\pi_{2,0}^*(\Gamma)]_{\widetilde{X}}$ is the covariant derivative, w.r.t. $\widetilde{X}$, of the pull-back connection $\pi_{2,0}^*(\Gamma)$. The previous definition is well posed as the vertical part of $\widetilde{X}$ gives no contribution. We call $\mathbf{II}$ the *universal second fundamental form* associated with Γ. This name is justified by the fact that, if $L \subset E$ is a submanifold, then the pull-back $(j^2 L)^* \mathbf{II}$ coincides with the second fundamental form on L. Moreover, we call the following map

$$\mathbf{H} \colon J^2(E, n) \to V_g^{1,0}, \quad \mathbf{H} \overset{\text{def}}{=} \frac{1}{n} \mathbf{II} \circ \bar{g}^H$$

the *universal mean curvature normal* (or *vector*) on n-dimensional submanifolds of E. It is easy to realize that both $\mathbf{II}^{-1}(0)$ and $\mathbf{H}^{-1}(0)$ are regular submanifolds of $J^2(E, n)$. They are the *totally geodesic submanifold equation* (see [19] for another geometric characterization of this equation) and the *minimal submanifold equation*. The coordinate expression of the latter is

$$(24) \quad \bar{g}^{H\lambda\xi} \left(u_{\lambda\xi}^k + \Gamma_\lambda{}^k{}_\xi + \Gamma_\lambda{}^k{}_i u_\xi^i + \Gamma_j{}^k{}_\xi u_\lambda^j + \Gamma_j{}^k{}_i u_\lambda^j u_\xi^i \right.$$
$$\left. - u_\beta^k \left(\Gamma_\lambda{}^\beta{}_\xi + \Gamma_\lambda{}^\beta{}_i u_\xi^i + \Gamma_j{}^\beta{}_\xi u_\lambda^j + \Gamma_j{}^\beta{}_i u_\lambda^j u_\xi^i \right) \right) = 0.$$

As an example, we write down the equation of minimal surfaces in the Euclidean space $\mathbb{R}^3$. We suppose that (x, y, u) is a Cartesian coordinate chart. Then we have

$$g^H = \begin{pmatrix} 1 + u_x^2 & u_x u_y \\ u_x u_y & 1 + u_y^2 \end{pmatrix}, \quad \bar{g}^H = \frac{1}{1 + u_x^2 + u_y^2} \begin{pmatrix} 1 + u_y^2 & -u_x u_y \\ -u_x u_y & 1 + u_x^2 \end{pmatrix}.$$

Hence the equation of totally geodesic submanifold is locally represented by the system $u_{xx} = 0$, $u_{xy} = 0$, $u_{yy} = 0$, and its contraction with $\bar{g}^H$ yields usual minimal surface equation $(1 + u_y^2) u_{xx} - 2 u_x u_y u_{xy} + (1 + u_x^2) u_{yy} = 0$. This coordinate representation does not cover the whole equation as a submanifold of $J^2(\mathbb{R}^3, 2)$. Two more divisions of the Cartesian chart (x, y, u) are needed, namely those for which x and y play the role of dependent variables, respectively.

The most important functional on submanifolds of a Riemannian manifolds is the area element

$$\mathcal{A}\colon J^1(E, n) \to \bigwedge^n (H^{1,0})^*, \quad \mathcal{A} = \sqrt{|g^H|}\, \overline{dx}^1 \wedge \cdots \wedge \overline{dx}^n,$$

where $|g^H| = \det((g^H)_{\lambda\mu})$. It is easy to realize that $(j^1 L)^* \mathcal{A}$ is the Riemannian area element on L on every chart. In general, the above functional is not global.

17 Theorem. *There are no global nowhere-vanishing Lagrangians on $J^1(E, 1)$.*

Proof. If such a Lagrangian exists, then the line bundle $H^{1,0}$ would be orientable. But $H^{1,0}$ is never orientable. In fact, if $H^{1,0}$ would be orientable then its pull-back on any fiber of $J^1(E, 1)$ would be orientable too. But this is a contradiction, because such a pull-back bundle is the canonical 1-vector bundle over the Grassmannian manifold $\mathrm{Gr}(T_p E, 1)$, that is not orientable [25]. □

Note that we could get a global area Lagrangian if we would use jets of *oriented* submanifolds. Anyway we do not need to postulate its globality in order to obtain global Euler–Lagrange equations. Let us denote by $\mathrm{Hess}(\mathcal{A})$ the (local) Hessian of $\mathcal{A}$, *i.e.*, the second differential of $\mathcal{A}$ along the fibers of $\pi_{1,0}$. We have

$$\mathrm{Hess}(\mathcal{A})\colon J^1(E, n) \to V^* J^1(E, n) \otimes V^* J^1(E, n) \otimes \bigwedge^n (H^{1,0})^*, \quad \mathrm{Hess}(\mathcal{A})^{\lambda\mu}_{ij} = \frac{\partial^2 \sqrt{|g^H|}}{\partial u^i_\lambda \partial u^j_\mu}$$

18 Theorem. *We have the equality*

$$\mathcal{E}(\mathcal{A}) = -\mathbf{II} \circ \mathrm{Hess}(\mathcal{A}), \quad where \quad \mathrm{Hess}(\mathcal{A})^{\lambda\mu}_{ij} = (\bar{g}^H)^{\lambda\mu}\, g^V_{ij}\, \sqrt{|g^H|}.$$

Proof. Lemma 4 and a long computation yields the result. We used (16) and the formulas

$$\frac{\partial \sqrt{|g^H|}}{\partial u^i_\lambda} = \frac{1}{2} \sqrt{|g^H|} (\bar{g}^H)^{\sigma\rho} \frac{\partial (g^H)_{\sigma\rho}}{\partial u^i_\lambda}, \quad \frac{\partial (\bar{g}^H)^{\rho\theta}}{\partial u^j_\mu} = -(\bar{g}^H)^{\rho\sigma} \frac{\partial (g^H)_{\sigma\nu}}{\partial u^j_\mu} (\bar{g}^H)^{\nu\theta}.$$

□

19 Corollary. *The Euler–Lagrange equation $\mathcal{E}(\mathcal{A}) = 0$ is an open submanifold of the minimal submanifold equation $\mathbf{H}^{-1}(0)$.*

4.2   Relativistic mechanics

Here we recall the geometric model for the phase space of one relativistic particle by Janyška and Modugno [10, 32]. The phase space in this model is an open submanifold of the first-order jet of curves in spacetime. We will show that the equation of particle motion presented in [10] is the Euler–Lagrange equation of a Lagrangian, following the scheme of section 3 (see [20] for more details).

Let us set $\dim E = 4$, and consider a *scaled* Lorentz metric

$$g \colon E \to \mathbb{L}^2 \otimes T^*E \otimes T^*E$$

on E, with signature $(+ - - -)$. Here $\mathbb{L}^2$ is the one-dimensional space of length units; we will also consider the mass and time one-dimensional spaces $\mathbb{M}$ and $\mathbb{T}$ (see [11] for more details on such spaces). The components $g_{\mu\nu}$ of g in local coordinates are $\mathbb{L}^2$-valued smooth functions on E. For physical reasons, we assume E to be oriented and time-like oriented. Latin indexes will label space-like coordinates, Greek indexes will label spacetime coordinates. We will use coordinate charts (x^0, x^i) such that $\partial/\partial x^0$ is time-like and time-like oriented, and $\partial/\partial x^i$ are space-like.

A *motion* is a time-like curve $s \subset E$, and its *velocity* is $j_1 s$. We consider the motion of a particle with mass $m \in \mathbb{M}$. We introduce the speed of light $c \in \mathbb{T}^{-1} \otimes \mathbb{L}$ and the Planck's constant $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$. The metric and the orientation yields a natural parallelization $Ts \simeq s \times \mathbb{T}$.

The set

$$U^1 E \subset J^1(E, 1)$$

of velocities of motions is said to be the *phase space*. By a restriction we have the natural projection $\pi_{1,0} \colon U^1 E \to E$. Time orientability implies that $H^{1,0} \simeq U^1 E \times \mathbb{T}$.

We introduce the normalized first-order contact map $Д \overset{\text{def}}{=} cD/\|D\|$, with coordinate expression $Д = c\alpha D_0$, where $\alpha \overset{\text{def}}{=} (g_{00} + 2g_{0j}x_0^j + g_{ij}x_0^i x_0^j)^{-1/2}$. The standard metric isomorphism yields a natural 1-form $\tau^\natural \overset{\text{def}}{=} g^\flat \circ Д$; in coordinates $\tau^\natural = \tau_\lambda dx^\lambda = c\alpha(g_{0\lambda} + g_{i\lambda}x_0^i)dx^\lambda$. Being $g(Д, Д) = c^2$, the inclusion $U^1 E \subset \mathbb{T}^* \otimes TE$ yields a non-linear bundle structure on $U^1 E$ with fibers diffeomorphic to $\mathbb{R}^3$.

After some intrinsic computations we obtain the *gravitational 2-form* $\Omega^\natural$ on $U^1 E$, with coordinate expression

$$\Omega^\natural = c\alpha(g_{i\mu} - c^{-2}\tau_i\tau_\mu)(dx_0^i - \Gamma_{\varphi 0}{}^i) \wedge dx^\mu,$$

where $\Gamma_{\varphi 0}{}^i = K_\varphi{}^i{}_j x_0^j + K_\varphi{}^i{}_0 - x_0^i(K_\varphi{}^0{}_j x_0^j + K_\varphi{}^0{}_0)$. It can be proved that $\Omega^\natural = d\tau^\natural$.

The *electromagnetic field* can be introduced as a scaled closed form F on E. It can be proved that the following *joined contact 2-form*

$$\Omega \overset{\text{def}}{=} m/\hbar \, \Omega^\natural + q/(2\hbar c) \, F$$

is non-degenerate. Its kernel is a foliation γ which yields the dynamics on the spacetime. If A is a local potential of F (according to $2dA = F$), then $m/\hbar \, \tau^\natural + q/(\hbar c) \, A$ is a local potential of Ω. Note that here $\hbar$ plays just the role of an overall scaling factor, and it has no influence on the equation of motion.

We obtain our results following the scheme of section 3, as we did in the previous example. First of all, we observe that

$$\lambda_{GR} \overset{\text{def}}{=} \left[\frac{m}{\hbar}\tau^\natural + \frac{q}{\hbar c}A\right] = h^{0,1}\left(\frac{m}{\hbar}\tau^\natural + \frac{q}{\hbar c}A\right)$$

is a first-order local Lagrangian on $U^1 E$, whose coordinate expression is

$$\lambda_{GR} = \left(\frac{mc}{\hbar} \sqrt{g_{00} + 2g_{0j} x_0^j + g_{ij} x_0^i x_0^j} + \frac{q}{\hbar c} \left(A_0 + x_0^i A_i \right) \right) \overline{dx}^0.$$

Then, the corresponding Euler–Lagrange morphism is

$$\tilde{e}_1(\lambda_{GR}) = \left[d \left(\frac{m}{\hbar} \tau^\natural + \frac{q}{\hbar c} A \right) \right] = [\Omega],$$

with coordinate expression

(25) $$\tilde{e}_1(\lambda_{GR}) = \frac{mc}{\hbar} \alpha (g_{ij} - c^{-2} \tau_i \tau_j)(x_{00}^i - (\gamma_{00}^i{}^\natural + \gamma_{00}^i{}^e)) \, \omega^j \otimes \overline{dx}^0,$$

where $\gamma_{00}^i{}^\natural + \gamma_{00}^i{}^e$ are the components of the above foliation γ, with coordinate expression

$$\gamma_{00}^i{}^\natural = K_0{}^i{}_0 - 2K_0{}^i{}_j x_0^j + K_0{}^0{}_0 x_0^i + 2K_0{}^0{}_j x_0^i x_0^j - K_j{}^i{}_k x_0^j x_0^k + K_j{}^0{}_k x_0^j x_0^k x_0^i,$$

$$\gamma_{00}^i{}^e = -\frac{q}{mc}(g^{i\mu} - x_0^i g^{0\mu})(F_{0\mu} + F_{j\mu} x_0^j).$$

Eq. (25) is equivalent to $x_{00}^i - (\gamma_{00}^i{}^\natural + \gamma_{00}^i{}^e) = 0$, and coincide with the equation of the integral curves of the foliation γ. This is the equation of particle motion in general relativity, written for non-parametrized time-like curves. Note that λ_{GR} has the same domain as A, hence it is global if and only if A is global (or vanishing).

20 Remark. The relativistic mechanics on jets of submanifolds has a distinguished feature: it can be easily proved that its Lagrangian is non-degenerate. In other words, $U^1 E$ is a natural first class Dirac constraint (see [20] for more details).

5   Conclusions

The fact that the objects in the $\mathcal{C}$-spectral sequence on submanifolds can be uniquely represented through the bundle of total derivatives allows us to consider a number of problems by analogy with the fibered case. Among such problems we have the classification of Lagrangians and/or Euler–Lagrange forms of a given order which are invariant under the action of a symmetry group or pseudogroup. In the Riemannian case, a first approach to the above problem is in [24], where isometry-invariant Lagrangians on jets of immersions are classified. Our language allows one to formulate the problem for more general objects, like Euler–Lagrange morphisms.

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